

Convergence of the Schwinger — DeWitt expansion for some potentials

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Running head: Convergence of the Schwinger — DeWitt
expansion . . .

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Abstract

It is studied time dependence of the evolution operator kernel for the Schrödinger equation with a help of the Schwinger — DeWitt expansion. For many of potentials this expansion is divergent. But there were established nontrivial potentials for which the Schwinger — DeWitt expansion is convergent. These are, e.g.,

$$V = g/x^2, \quad V = -g/\cosh^2 x, \quad V = g/\sinh^2 x, \quad V = g/\sin^2 x.$$

For all of them the expansion is convergent when $g = \lambda(\lambda - 1)/2$ and λ is integer. The theories with these potentials have no divergences and in this meaning they are "good" potentials contrary to other ones. So, it seems natural to pay special attention namely to these "good" potentials. Besides convergence they have other interesting feature: convergence takes place only for discrete values of the charge g . Hence, in the theories of this class the charge is quantized.

Key words

Asymptotic expansions, Schrödinger equation, evolution operator kernel, quantization of charge.

1 General relations

Various approaches in the quantum theory use the short-time Schwinger — DeWitt expansion [Schwinger 1951, DeWitt 1965, 1975, Osborn 1983, Barvinsky 1995]. So as other expansions in different parameters: conventional perturbation theory [Bender 1969, Lipatov 1977], the WKB-expansion, $1/n$ -expansion [Popov 1992] etc, it is usually treated as asymptotic one. Because of divergence of these expansions many difficulties in the quantum theory arises. Particularly, in QCD one cannot obtain correct predictions for low energy phenomena and so on.

Now we consider one interesting phenomenon, related to the Schwinger — DeWitt expansion, which, may be, will allow us in the future to take of some problems arising from divergence of the series.

The Schwinger — DeWitt expansion is specific representation of solution of the following problem for the evolution operator kernel

$$i \frac{\partial}{\partial t} \langle q', t | q, 0 \rangle = -\frac{1}{2} \frac{\partial^2}{\partial q'^2} \langle q', t | q, 0 \rangle + V(q') \langle q', t | q, 0 \rangle, \quad (1)$$

with initial condition

$$\langle q', t = 0 | q, 0 \rangle = \delta(q' - q). \quad (2)$$

The kernel $\langle q', t | q, 0 \rangle$ is written as

$$\langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi i t}} \exp \left\{ i \frac{(q' - q)^2}{2t} \right\} F(t; q', q), \quad (3)$$

and F according to [Schwinger 1951, DeWitt 1965, 1975, Slobodenyuk 1993, 1995 [10]] is expanded in powers of t

$$F(t; q', q) = \sum_{n=0}^{\infty} (it)^n a_n(q', q). \quad (4)$$

Here and everywhere below dimensionless values defined in obvious manner are used. The potential $V(q)$ is continuous function.

It is easy to derive from (1)–(3) the problem for the function F . The latter should satisfy the equation

$$i \frac{\partial F}{\partial t} = -\frac{1}{2} \frac{\partial^2 F}{\partial q'^2} + \frac{q' - q}{it} \frac{\partial F}{\partial q'} + V(q') F \quad (5)$$

and initial condition

$$F(t = 0; q', q) = 1. \quad (6)$$

The coefficient functions $a_n(q', q)$ may be determined from the sequence of recurrent relations

$$a_0(q', q) = 1, \quad (7)$$

$$n a_1(q', q) + (q' - q) \frac{\partial a_1(q', q)}{\partial q'} = a_1(q', q') = -V(q'), \quad (8)$$

for $n > 1$

$$n a_n(q', q) + (q' - q) \frac{\partial a_n(q', q)}{\partial q'} = \frac{1}{2} \frac{\partial^2 a_{n-1}(q', q)}{\partial q'^2} - V(q') a_{n-1}(q', q). \quad (9)$$

Eqs. (7)–(9) show that a_n can be calculated via a_{n-1} by means of integral relations

$$a_n(q', q) = \int_0^1 \eta^{n-1} d\eta \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} - V(x) \right\} a_{n-1}(x, q) \Big|_{x=q+(q'-q)\eta}. \quad (10)$$

Combinations of Eqs. (10) for different numbers n allows us to represent a_n for given n through the potential V and its derivatives

$$\begin{aligned} a_n(q', q) = & - \int_0^1 \eta_n^{n-1} d\eta_n \int_0^1 \eta_{n-1}^{n-2} d\eta_{n-1} \dots \int_0^1 \eta_2 d\eta_2 \int_0^1 d\eta_1 \times \\ & \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_n^2} - V(x_n) \right\} \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_{n-1}^2} - V(x_{n-1}) \right\} \dots \\ & \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_2^2} - V(x_2) \right\} V(x_1). \end{aligned} \quad (11)$$

Here $x_i = q + (x_{i+1} - q)\eta_i$, $x_{n+1} = q'$. Derivatives with respect to different x_i may be easily connected with each other

$$\frac{\partial}{\partial x_i} = \eta_{i-1} \frac{\partial}{\partial x_{i-1}} = \eta_{i-1} \eta_{i-2} \frac{\partial}{\partial x_{i-2}} \quad (12)$$

etc.

Other useful representation for the kernel may be obtained if in some domain for potential $V(q)$ the Taylor expansion takes place

$$V(q') = \sum_{k=0}^{\infty} \Delta q^k \frac{V^{(k)}(q)}{k!} \quad (13)$$

(here $V^{(k)}(q)$ is k th derivative of $V(q)$, $\Delta q = q' - q$), then the following representation of F may be used in calculations

$$F(t; q', q) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (it)^n \Delta q^k b_{nk}(q). \quad (14)$$

For the coefficients b_{nk} one has algebraic recurrent relations:

$$b_{1k} = - \frac{V^{(k)}(q)}{(k+1)!}, \quad (15)$$

$$b_{nk} = \frac{1}{n+k} \left[\frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \sum_{m=0}^k \frac{V^{(m)}(q)}{m!} b_{n-1,k-m} \right]. \quad (16)$$

Both representations (11) and (14)–(16) are used for analysis of convergence of the Schwinger — DeWitt expansion (behaviour of a_n for $n \rightarrow \infty$) in general case and for specific potentials.

This formalism can be easily modified for application to singular potentials with singularity of type $1/q^2$ at $q = 0$ [Slobodenyuk 1996 [12]]. One should take instead of initial condition (2) the following one

$$\langle q', t = 0 \mid q, 0 \rangle = \delta(q' - q) + A\delta(q' + q) \quad (17)$$

which may provide fulfillment of boundary condition for the wave function $\psi(q)$ at $q = 0$ ($\psi(q)$ should vanish at $q = 0$) by appropriate choice of constant A . Constant A is determined by requirement that the kernel does not have singularity at $q = 0$ or $q' = 0$ ($t \neq 0$). In correspondence with (17) the kernel is represented through two functions $F^{(\pm)}$ as

$$\begin{aligned} \langle q', t \mid q, 0 \rangle = & \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' - q)^2}{2t} \right\} F^{(-)}(t; q', q) + \\ & A \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' + q)^2}{2t} \right\} F^{(+)}(t; q', q), \end{aligned} \quad (18)$$

where $F^{(\pm)}$ can be expanded analogously to (4).

Generalization on the three-dimensional case is obvious. So we will not discuss it specially.

2 Examples of convergent expansions

It was shown in [Slobodenyuk 1995 [10]] that $|a_n(q', q)| \sim \Gamma(bn)$, where $0 < b \leq 1$, if there is no any cancellations between different contributions into a_n . For the most number of the potentials such factorial growth really takes place. Namely, for $V(q)$ represented by polynomial of order L constant b is $b = (L - 2)/(L + 2)$, for other $V(q)$ constant b is equal to 1. So, the Schwinger — DeWitt expansion factorially diverges and the point $t = 0$ is essential singular point of the function F . But there exist some kinds of potentials, for which cancellations are so essential that the series (4) is

convergent, and F is analytic at the point $t = 0$. This takes place only for some discrete values of charge. The examples of such potentials were presented in [Slobodenyuk 1995 [11], 1996 [13]]

$$V(q) = -\frac{g}{\cosh^2 q}, \quad (19)$$

$$V(q) = \frac{g}{q^2}, \quad (20)$$

$$V(q) = \frac{g}{\sinh^2 q}, \quad (21)$$

$$V(q) = \frac{g}{\sin^2 q}, \quad (22)$$

$$V(q) = aq^2 + \frac{g}{q^2}. \quad (23)$$

For all of them the expansion is convergent when $g = \lambda(\lambda - 1)/2$ and λ is integer.

For illustration we consider the potential (20). This potential has singularity at $q = 0$, so special formalism should be applied here. But for the sake of brevity we calculate only the function $F^{(-)}$, which is denoted simply as F .

Expansion (13) for the potential (20) has the finite convergence range $R(q) = q$, finiteness of which is connected with singularity of $V(q)$ at the point $q = 0$. The derivatives $V^{(k)}$ may be easily calculated

$$V^{(k)}(q) = (-1)^k \frac{\lambda(\lambda - 1)}{2} \frac{(k + 1)!}{q^{k+2}}. \quad (24)$$

Substituting (24) into (15), (16) and diminishing n times the first index of $b_{nk}^{(-)}$ by means of (16) we get

$$\begin{aligned} b_{nk}^{(-)} &= \frac{(-1)^{n+k}}{q^{2n+k}} \frac{(k + n - 1)!}{n!(n - 1)!k!} \prod_{j=1}^n \left(\frac{\lambda(\lambda - 1)}{2} - \frac{j(j - 1)}{2} \right) \\ &= \frac{(-1)^{n+k}}{q^{2n+k}} \frac{(k + n - 1)!}{n!(n - 1)!k!} \frac{\Gamma(\lambda + n)}{2^n \Gamma(\lambda - n)}. \end{aligned} \quad (25)$$

It is obvious that if λ is noninteger then $|b_{nk}^{(-)}| \sim n!$ for $n \rightarrow \infty$. So, for noninteger λ expansion (14) for potential (20) is divergent. But if λ is

integer ($\lambda > 1$, because cases $\lambda = 0$, $\lambda = 1$ are trivial) then one can easily see from (25) that only the coefficients $b_{nk}^{(-)}$ for $n < \lambda$ are different from zero, and in (14) the series in powers of t is really the polynomial of finite degree $\lambda - 1$. Let us substitute (25) into (14) and make summation over k . Then we get finally

$$F^{(-)}(t; q', q) = 1 + \sum_{n=1}^{\infty} \left(\frac{-it}{2q'q} \right)^n \frac{\Gamma(\lambda + n)}{n! \Gamma(\lambda - n)}. \quad (26)$$

The series in (26) has the following feature: if λ is noninteger, then coefficient in front of t^n grows as $n!$ for $n \rightarrow \infty$ and the series is asymptotic; but if λ is integer, then series contains only finite number of terms (sum is made really till $n = \lambda - 1$). So, in latter case the expansion is convergent.

Singularities of $F^{(\pm)}$ at $q = 0$ cancel each other and the kernel $\langle q', t | q, 0 \rangle$ has no such singularity. Substitution of obtained expressions for $F^{(\pm)}$ into (18) gives us asymptotic expansion of the function

$$\langle q', t | q, 0 \rangle = e^{-i\frac{\pi}{2}(\lambda-1/2)} \frac{\sqrt{q'q}}{it} \exp \left\{ i \frac{q'^2 + q^2}{2t} \right\} J_{\lambda-1/2} \left(\frac{q'q}{t} \right) \quad (27)$$

for small t (large $q'q/t$). Eq. (27) coincides with known expression derived by other methods. Asymptotic expansion for the Bessel function is not divergent only for semi-integer order, i.e., for integer λ . In this case the series contains finite number of terms and point $t = 0$ is regular point contrary to the case of noninteger λ , when $t = 0$ is essential singular point.

For other potentials of the series (19)–(23) the situation is similar. The expansion (4) is convergent only for integer λ and divergent in other case (but it includes infinite number of terms).

Moreover, careful study shows us that if we consider the coupling constant g of continuous potential $V(q)$ as independent variable, then the coefficients a_n of representation (3)–(4) for the evolution operator kernel increase for $n \rightarrow \infty$ as

$$a_n \sim \Gamma \left(n \frac{L-2}{L+2} \right)$$

for the potentials being expressed via the polynomial of order L and as

$$a_n \sim n!$$

for other potentials. So, the Schwinger — DeWitt expansion in this supposition is divergent for all potentials excluding polynomials of order not higher than two.

If the charge is treated as fixed parameter, then because of cancellations for some kinds of the potentials and for some values of the charge g expansion (4) may be convergent. Examples of such potentials are presented above. Discreteness of the charge for the class of the potentials for which the Schwinger — DeWitt expansion is convergent, probably, may be connected with discreteness of the charge in the nature. In this correspondence, the potentials of this class are of special interest. Operating with them we get rid of some kind of divergences in the theory and, at the same time, have a theory with discrete charge. So, it is necessary to look for other potentials of this class and study them carefully. Besides, it is interesting to propagate such analysis on the quantum field theory. One may hope that it will allow us to reconstruct quantum electrodynamics with exactly fixed charge e .

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